JOURNAL OF GUIDANCE, CONTROL, AND DYNAMICS
Vol. 29, No. 3, May–June 2006

Closed-Form Solutions for Near-Circular Arcs with Quadratic Drag

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This paper considers the restricted two-body problem in the presence of drag that varies with a power of the magnitude of the velocity. In general, the orbit equation for this problem is an integrodifferential equation. For extreme cases in which the motion is mostly tangential and the drag varies with the square of the magnitude of the velocity, a new transformation of the orbit equation results in an ordinary differential equation. For a model atmosphere in which the density varies inversely as the square of the distance from the center of attraction, we provide closed-form solutions to this differential equation. This extends previous work in which the density of the model atmosphere varies inversely with the distance from the center of attraction.

I. Introduction

A CLASSICAL problem treats the motion of a particle in a central force field[1]

$$\ddot{\boldsymbol{R}} = -f(R)\boldsymbol{R} \tag{1}$$

where $\boldsymbol{R}$ is the radius vector from the center of attraction in three-dimensional Euclidean space $R = |\boldsymbol{R}| = (\boldsymbol{R}\cdot\boldsymbol{R})^{1/2}$, and differentiation with respect to time, is represented by a dot. Some results on the related problems of relative motion and terminal rendezvous were found[2] for this general central force field. For the case of a Newtonian gravitational force field, there is need for a model that includes the effect of atmospheric drag and that is amenable to an analytic solution. Several solutions have been presented in which the drag is linear in the velocity.[3–6] This kind of model has also been used for relative motion studies.[7] Relatively little has been found in the literature on closed-form solutions of the equations of motion with quadratic drag. There have been analytical approaches via perturbations that consider both quadratic drag and earth oblateness effects. Some early approaches of this type are summarized in the work of Lane and Cranford.[8] Later approaches of this type are found in the work of Hoots and France,[9] King-Hele,[10] and in Vallado's book.[11]

Recently the authors have had some success with certain limiting assumptions in finding analytic solutions in which the drag varies with the square of the magnitude of the velocity.[12] This approach also applies to relative motion studies.[13]

This previous work considered a model of the form

$$\ddot{\boldsymbol{R}} = -f(R)\boldsymbol{R} - \rho(\alpha, R)(\dot{\boldsymbol{R}}\cdot\dot{\boldsymbol{R}})^n\dot{\boldsymbol{R}} \tag{2}$$

where $f(R) = \mu/R^3$, $\rho(\alpha, R) = \alpha/R$, $n = \frac{1}{2}$, μ is the product of the universal gravitational constant and the central mass, and α is a parameter representing the drag coefficient and geometry of the satellite and is usually considered constant over intervals in which R changes little.

In one of the results of this work, we simplified Eq. (2) under the assumption that the radial speed is small compared with the tangential speed and found a closed-form solution for the orbit equation. In the present paper we present another novel transformation of the orbit equation that leads to a closed-form solution for the case $\rho(\alpha, R) = \alpha/R^2$.

We begin with consideration of the motion of a particle under the more general form

$$\ddot{\boldsymbol{R}} = -F(\boldsymbol{R}, \dot{\boldsymbol{R}})\boldsymbol{R} - G(\alpha, \boldsymbol{R}, \dot{\boldsymbol{R}})\dot{\boldsymbol{R}} \tag{3}$$

and, without additional difficulty, derive the more general orbit equation as an integrodifferential equation. In this context we note that the right sides of Eqs. (2) and (3) can be viewed as a noncentral-force field. The treatment of motions under the influence of noncentral potentials has been the subject of numerous papers. For a survey and references, see Kaushal's book.[14]

We then present a novel transformation that for quadratic drag and motion, which is mostly tangential, converts the integrodifferential equation for the orbit to an ordinary differential equation that applies to a class of atmospheric density models. For the model in which the atmospheric density varies with $1/R^2$, a closed-form solution is presented. This augments previous work[12] of the authors in which the atmospheric density was assumed to vary with $1/R$.

Closed-form solutions have also been found by the authors for the case of motion that is mostly radial.[15,16] Because this case is of less importance, it is not included in this paper.

A discussion of the motion of a particle satisfying Eq. (3) and a derivation of the orbit equation are found in Sec. II. In Sec. III we investigate this equation for a power-law drag force, then simplify for the quadratic drag model under consideration. In the extreme case in which the motion is mostly tangential, the orbit equation can be approximated and simplified. We then derive orbital conditions under which this approximation applies. In Sec. IV we find closed-form solutions for the simplified equations. Section V contains our conclusions.

II. Reduction of the Equations of Motion

Taking the vector product of Eq. (3) with $\boldsymbol{R}$ from the left and introducing the specific angular momentum $\boldsymbol{L} = \boldsymbol{R}\times\dot{\boldsymbol{R}}$, we obtain

$$\dot{\boldsymbol{L}} + G(\alpha, \boldsymbol{R}, \dot{\boldsymbol{R}})\boldsymbol{L} = 0 \tag{4}$$

hence,

$$\boldsymbol{L}\times\dot{\boldsymbol{L}} = 0 \tag{5}$$

Received 19 February 2005; revision received 28 April 2005; accepted for publication 29 April 2005; presented as Paper 2004-5302 at the AIAA/AAS Astrodynamics Specialist Conference, Providence, RI, 16–19 August 2005.

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It follows then that $\dot{\boldsymbol{L}}$ is always parallel to $\boldsymbol{L}$; consequently, $\boldsymbol{L}$ has a constant direction. This shows that the motion under consideration takes place in a fixed plane. We therefore introduce polar coordinates R, θ in this plane. Rewriting the equation of motion (3) in terms of polar coordinates we then have

$$R\ddot{\theta} + 2\dot{R}\dot{\theta} = -G(\alpha, R, \dot{R}, \theta, \dot{\theta})R\dot{\theta} \tag{6}$$

$$\ddot{R} - R\dot{\theta}^2 = -F(R, \dot{R}, \theta, \dot{\theta})R - G(\alpha, R, \dot{R}, \theta, \dot{\theta})\dot{R} \tag{7}$$

Multiplying Eq. (6) by R, we obtain

$$\frac{\mathrm{d}(R^2\dot{\theta})}{\mathrm{d}t} = -G(\alpha, R, \dot{R}, \theta, \dot{\theta})R^2\dot{\theta} \tag{8}$$

Dividing by $R^2\dot{\theta}$ and integrating, we see that

$$R^2\dot{\theta} = J \tag{9}$$

where

$$J = \exp\left[-\int G(\alpha, R, \dot{R}, \theta, \dot{\theta})\,\mathrm{d}t\right] \tag{10}$$

(Here and throughout the paper we use indefinite integrals for ease of the analysis.) It follows from Eqs. (9) and (10) that $\dot{\theta}$ is identically positive unless the motion is rectilinear. Because we are interested in motion that is mostly tangential, we shall not consider this special case. Using Eq. (9) to make a change of the independent variable from t to θ in Eq. (7), we obtain after some algebra the following integrodifferential equation for the orbit:

$$R\frac{\mathrm{d}^2 R}{\mathrm{d}\theta^2} - 2\left(\frac{\mathrm{d}R}{\mathrm{d}\theta}\right)^2 = R^2 - \frac{F(R, \dot{R}, \theta, \dot{\theta})R^6}{\mathrm{J}^2} \tag{11}$$

where $R, \dot{R}, \dot{\theta}$, and J are now regarded as functions of θ.

In this expression the effect of the noncentral force is contained in J. We would like to further reduce Eq. (11) to an ordinary differential equation and find examples for which there are closed-form solutions. As we shall see in the next section, we can apply an approximation used in a previous work[12] that will accomplish this for the quadratic drag model presented here. We note however that in some cases it might be advantageous to change the integration variable in Eq. (10) from t to θ or R or to use another integration scheme to express $R^2\dot{\theta}$.

III. Power-Law Drag and Helpful Approximation for Motion That Is Mostly Tangential

For the case of a power-law drag,

$$G(\alpha, R, \dot{R}, \theta, \dot{\theta}) = \rho(\alpha, R)(\dot{\boldsymbol{R}} \cdot \dot{\boldsymbol{R}})^n \tag{12}$$

We now show that for nearly tangential motion where $|\dot{R}| \ll |R\dot{\theta}|$ one can approximate Eq. (11), then solve it analytically for Newtonian gravitation and the drag model that we present. At the end of this section, we show, in general, where this approximation is reasonable.

This special case occurs if the motion is nearly all tangential, as is the case for low-eccentricity elliptical orbits, or for restricted regions near the apses in more highly elliptical orbits. In this case we write

$$(\dot{\boldsymbol{R}} \cdot \dot{\boldsymbol{R}})^n = (\dot{R}^2 + R^2\dot{\theta}^2)^n = (R^2\dot{\theta}^2)^n[1 + (\dot{R}/R\dot{\theta})^2]^n \tag{13}$$

and, because of the condition $|\dot{R}| \ll |R\dot{\theta}|$, we linearize the right-hand side of Eq. (13) and obtain the approximation

$$(\dot{\boldsymbol{R}} \cdot \dot{\boldsymbol{R}})^n = (R\dot{\theta})^{2n} \tag{14}$$

It follows that Eq. (12) takes the form

$$G(\alpha, R, \dot{R}, \theta, \dot{\theta}) = \rho(\alpha, R)(R\dot{\theta})^{2n} \tag{15}$$

and Eq. (8) becomes

$$\frac{\mathrm{d}(R^2\dot{\theta})}{\mathrm{d}t} = -\rho(\alpha, R)R(R\dot{\theta})^{2n+1} \tag{16}$$

leading to a corresponding change in Eq. (10).

A. Quadratic Drag

In the following we shall show that for $n = \frac{1}{2}$ (quadratic drag) and the aforementioned special case, the integrodifferential equation (11) can be reduced to a differential equation. The reduction is demonstrated for atmospheric density models that vary inversely with a power of R.

Under the condition $n = \frac{1}{2}$, Eq. (16) becomes

$$\frac{\mathrm{d}(R^2\dot{\theta})}{\mathrm{d}t} = -\rho(\alpha, R)R^3\dot{\theta}^2 \tag{17}$$

hence,

$$R^2\dot{\theta} = \exp\left[-\int \rho(\alpha, R)R\,\mathrm{d}\theta\right] \tag{18}$$

We now show that if $F = F(R, R', \theta)$, where R' denotes differentiation with respect to θ and the function $\rho(\alpha, R)R$ is invertible, then Eq. (11) can be reduced to an ordinary differential equation. To accomplish this, we introduce the transformation

$$u(\theta) = \int \rho(\alpha, R)R\,\mathrm{d}\theta \tag{19}$$

We then have

$$u' = \rho(\alpha, \mathrm{R})\mathrm{R} \tag{20}$$

$$u'' = [\rho'(\alpha, R)R + \rho(\alpha, R)]R' \tag{21}$$

$$u''' = [\rho'(\alpha, R)R + \rho(\alpha, R)]R'' + [\rho''(\alpha, R)R + 2\rho'(\alpha, R)](R')^2 \tag{22}$$

By the invertibility assumption, we can solve Eq. (20) for R in terms of u', then Eq. (21) can be solved for R' in terms of u' and u''. Finally, using Eq. (22), R'' can be expressed in terms of u', u'', and u'''. Substituting these expressions for R, R', and R'', Eq. (11) reduces to a nonlinear third-order differential equation in $u(\theta)$ because $J = e^{-u}$.

We carry out this procedure explicitly for the special case, where

$$\rho(\alpha, \mathrm{R}) = \alpha/\mathrm{R}^m \qquad m = 2, 3, \ldots \tag{23}$$

This extends previous work,[12] which treats the case $m = 1$.

Instead of defining u by Eq. (19), we make a minor modification, and, for expediency, we define u without α as

$$u = \int \frac{\mathrm{d}\theta}{\mathrm{R}^{m-1}} \tag{24}$$

Rewriting Eq. (11) as

$$\mathrm{R}''/\mathrm{R} - 2(\mathrm{R}'/\mathrm{R})^2 = 1 - F(\mathrm{R}, \mathrm{R}', \theta)\mathrm{R}^4/\mathrm{J}^2 \tag{25}$$

evaluating u', u'', and u''' from Eq. (24), and substituting into Eq. (25), we obtain

$$u'''/(m-1)u' - [(m-2)/(m-1)^2](u''/u')^2 = -1 + F(R, R', \theta)R^4 e^{2\alpha u} \tag{26}$$

where on the right-hand side R and R' have to be replaced by their proper expressions in terms of u' and u''.

For the Newtonian gravitational case where $F = \mu/R^3$, we then have

$$u'''/(m-1)u' - [(m-2)/(m-1)^2](u''/u')^2 = -1 + \mu e^{2\alpha u}(u')^{-1/(m-1)} \tag{27}$$

This ordinary differential equation is third order in u. It can be reduced to second order if we introduce

$$p = \frac{\mathrm{d}u}{\mathrm{d}\theta} \tag{28}$$

as the dependent variable and consider u as the independent one. With this change of variables, Eq. (27) becomes

$$\frac{p^{1/(m-1)}}{m-1}\left[\frac{\mathrm{d}}{\mathrm{d}u}\left(p\frac{\mathrm{d}p}{\mathrm{d}u}\right) - \frac{m-2}{m-1}\left(\frac{\mathrm{d}p}{\mathrm{d}u}\right)^2\right] = -p^{1/(m-1)} + \mu e^{2\alpha u} \tag{29}$$

Finally if we introduce the variable

$$v = p^{m/(m-1)} \tag{30}$$

then Eq. (29) becomes

$$\frac{1}{m}\frac{\mathrm{d}^2 v}{\mathrm{d}u^2} + v^{(2-m)/m} = \mu e^{2\alpha u} v^{-[(m-1)/m]} \tag{31}$$

We observe that if a solution $v = v(u)$ of this equation is known then Eqs. (30) and (28) provide a solution for $u'(\theta)$. Because $u' = 1/R^{m-1}$, one can solve for $R = R(\theta)$.

In spite of the fact that Eqs. (27) and (31) are nonlinear, we can make further progress for $m = 2$. In this special case these equations respectively reduce to

$$u''' + u' - \mu e^{2\alpha u} = 0 \tag{32}$$

$$\frac{1}{2}\frac{\mathrm{d}^2 v}{\mathrm{d}u^2} + 1 = \mu e^{2\alpha u} v^{-\frac{1}{2}} \tag{33}$$

B. Regions Where the Approximation Is Reasonable

We now describe conditions under which the just-mentioned approximation is reasonable. If a satellite is in Keplerian orbit,

$$R = P/[1 + \epsilon \cos(\theta - \theta_0)] \tag{34}$$

where P is the semiparameter, ϵ the eccentricity, θ_0 the argument of perigee, and θ is the true latitude (i.e., sum of the argument of perigee and true anomaly). Differentiating with respect to time, we find

$$\dot{R} = \frac{\epsilon \sin(\theta - \theta_0) R\dot{\theta}}{1 + \epsilon \cos(\theta - \theta_0)} \tag{35}$$

We assume that the approximation is reasonable for ϵ and $\theta - \theta_0$ where $|\dot{R}|R\dot{\theta}|$ is small. This analysis is not rigorous because the atmospheric drag is not included in Eqs. (34) and (35). The advantage of this analysis is in its simplicity. Its usefulness increases as the altitude becomes higher and the effect of drag becomes smaller.

We are assuming that $|\dot{R} \ll |R\dot{\theta}|$. Letting M denote a small positive number, we seek conditions under which

$$\left|\frac{\dot{R}}{R\dot{\theta}}\right| = \frac{\epsilon|\sin(\theta - \theta_0)|}{1 + \epsilon\cos(\theta - \theta_0)} \leq \mathrm{M} \tag{36}$$

Figure 1 shows polar plots of R/P from Eq. (34) for $\theta_0 = 0$ and $\epsilon = 0.1$, 0.3, and 0.4. Setting $M = 0.05$, the regions near the apses where Eq. (36) is satisfied are highlighted by a bold arc. The bold arcs indicate regions where the approximation Eq. (14) is reasonable. To be more conservative, one could use a smaller number for M.

Solving the inequality (36) for ϵ, we obtain

$$\epsilon \leq M/[|\sin(\theta - \theta_0)| - M\cos(\theta - \theta_0)] \tag{37}$$

where the denominator is positive. If the denominator is negative or zero, then any nonnegative value of ϵ satisfies the inequality (36). It follows that if $|\tan(\theta - \theta_0)| \leq M$, then any nonnegative value of ϵ satisfies inequality (36). On the other hand, if $\epsilon \leq M$ then any value of $\theta - \theta_0$ satisfies inequality (36). This indicates that the approximation is reasonable for near-circular orbits, that is, $\epsilon \leq M$.

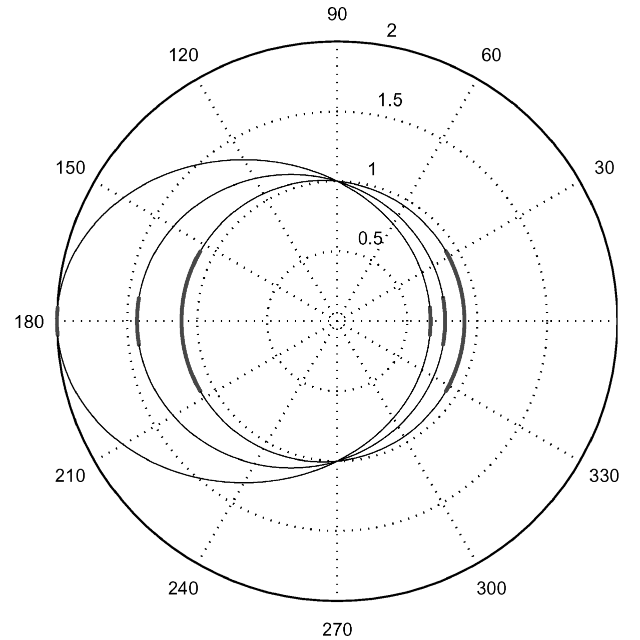

Fig. 1 Elliptical orbits with arcs highlighted where $|\dot{R}/R\dot{\theta}| \leq 0.05$ for eccentricities of 0.1, 0.3, and 0.4.

The inequality (37) can be represented by the region on or below the curve presented in Fig. 2, where for simplicity we have set $\theta_0 = 0$. For this example true latitude and true anomaly are identical. On the left is a vertical asymptote at $\theta = \tan^{-1} M$. The curve has a relative maximum at $\theta = \pi$, where $\epsilon = 1$, and is open on the right as ϵ approaches 1.

IV. Closed-Form Solutions of the Orbit Equation for $\rho(\alpha, R) = \alpha/R^2$

We find an example in which Eq. (11) can be approximated and solved in closed form. In this example the flight time t can be represented by an integral in terms of θ. This example is roughly applicable on the arcs satisfying inequality (37). The only other closed-form solutions found for this class of problems[12] were found for $\rho(\alpha, R) = \alpha/R$. In this new model,

$$\rho(\alpha, R) = \alpha/R^2 \tag{38}$$

Recall that we are considering a case where the radial speed is relatively small, $|\dot{R}| \ll |R\dot{\theta}|$. It is seen in the preceding section that this model leads to either Eq. (32) or (33), which are nonlinear. In view of Eq. (24), these equations can be simplified for θ intervals not exceeding one or two revolutions and high altitudes where the constant α is very small. This simplification is applicable for many more revolutions if the initial orbit is nearly circular. For these types of applications, $|\alpha u| \ll 1$, and Eq. (32) can be approximated by the linear equation

$$u''' + u' - \mu(1 + 2\alpha u) = 0 \tag{39}$$

A. Solution of the Equation

The characteristic equation associated with the homogeneous form of Eq. (39) is

$$\lambda^3 + \lambda - 2\mu\alpha = 0 \tag{40}$$

This equation has one positive root

$$\lambda = a > 0 \tag{41}$$

and two complex roots

$$\lambda = \sigma \pm \omega i \tag{42}$$

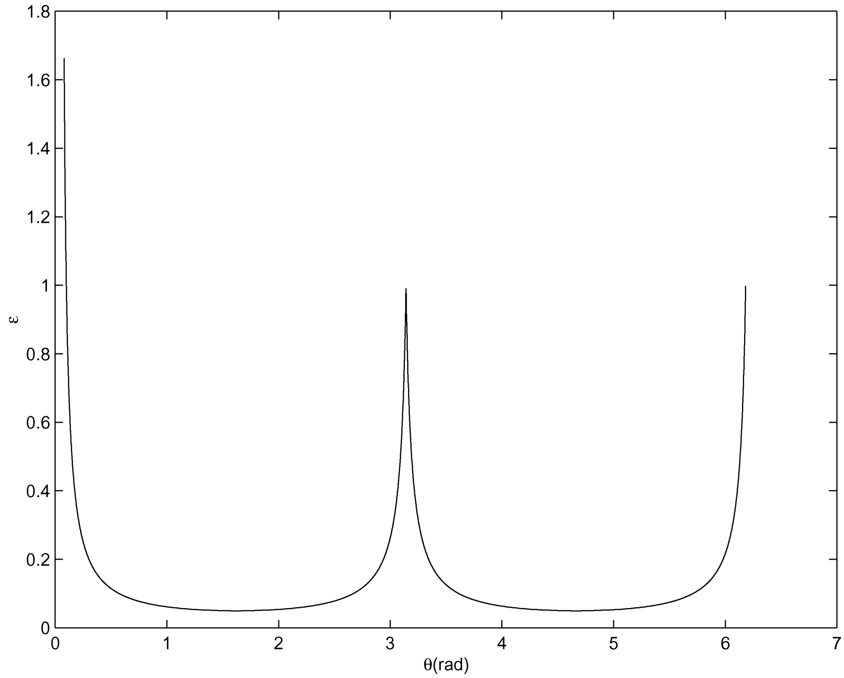

Fig. 2 Eccentricity vs true anomaly where $|\dot{R}/R\dot{\theta}| = 0.05$.

Solving the cubic equation (40), we find that

$$a = \beta - 1/3\beta \tag{43}$$

where

$$\beta = \left[\frac{(27\mu^2\alpha^2 + 1)^{\frac{1}{2}}}{3\sqrt{3}} + \mu\alpha\right]^{\frac{1}{3}} \tag{44}$$

$$\sigma = -\frac{a}{2} \tag{45}$$

and

$$\omega = (4 + 3a^2)^{\frac{1}{2}}\big/2 \tag{46}$$

For this reason the complete solution of Eq. (39) can be written in the form

$$u = -1/2\alpha + c_1 e^{a\theta} + c_2 e^{-(a/2)\theta}\cos\omega(\theta - \phi) \tag{47}$$

where c_1, c_2, and ϕ are arbitrary constants. Differentiating this expression, we obtain

$$u' = ac_1 e^{a\theta} + c_2 \nu e^{-a\theta/2}\cos\omega(\theta - \theta_0) \tag{48}$$

where $\nu = \sqrt{(1 + a^2)}$ and $\theta_0 = \phi + (1/\omega)\tan^{-1}[\sqrt{(4 + 3a^2)}/a]$. Recalling from Eq. (24) that $u' = 1/R$, we can write the solution of the orbit equation in the convenient form

$$R = \frac{Pe^{-a\theta}}{1 + \epsilon e^{-3a/2}\cos\omega(\theta - \theta_0)} \tag{49}$$

where $P = 1/ac_1$ and $\epsilon = \nu c_2/ac_1$. This result should be compared with Eq. (38) of Ref. 12, which arose from the assumption that $\rho(\alpha, R) = \alpha/R$. Our new result has a higher orbital frequency as is evident from the factor ω in the formula. If $\epsilon = 0$, then Eq. (49) simplifies

$$R = Pe^{-a\theta} \tag{50}$$

The form of this result is similar to the analogous result from Refs. 12 and 13. In these references it is stated that the condition $\epsilon = 0$ results from initial conditions that would produce an orbit that would be circular without drag. The statements in these references are incorrect, although the orbit is indeed very nearly circular. Differentiating Eq. (50), we obtain

$$R'(0) = -aP \tag{51}$$

where $P = R(0)$. Because a is very small but not zero, we see that this fictitious orbit is very nearly circular, but not a circle. We note that the curve defined by Eq. (49) converges to the one defined by Eq. (50).

Equation (50) provides a method of calculating the drag constant α from the decay in the altitude of the orbit it defines. If $R_0 = R(0)$ is an initial radial distance from the center of attraction, after the radial distance has decayed to $R = R(\theta)$, then

$$a = \ln(R_0/R)/\theta \tag{52}$$

The drag constant α can then be calculated via Eq. (40):

$$\alpha = a(1 + a^2)/2\mu \tag{53}$$

The time in orbit can be found in terms of an integral. Indeed, Eq. (18) becomes

$$R^2\dot{\theta} = e^{-\alpha u} \tag{54}$$

Separating variables provides the following integral for time in orbit:

$$t = \int \frac{e^{\alpha u(\theta)}\,\mathrm{d}\theta}{u'(\theta)^2} \tag{55}$$

where u and u' are given respectively by Eqs. (47) and (48). If the orbit is defined by Eq. (50), then $c_2 = 0$, and the expression simplifies upon introducing the variable $v = e^{a\theta}$:

$$t = \frac{P^2}{e^{\frac{1}{2}}a}\int \frac{e^{c_1\alpha v}}{v^3}\,\mathrm{d}v \tag{56}$$

An approximation can be found by replacing the exponential by a Taylor polynomial.

A numerical check on the accuracy of the approximation of Eq. (32) by Eq. (39) was performed. The approximation is found to be highly accurate.

A polar plot of R/P from the approximate equation (49) is presented in Fig. 3 for $\theta_0 = 0$, $\epsilon = 0.25$, and $a = 0.01$. One can see that the curve tends to circularize. After that the curve behaves more like the exponential curve of Eq. (50). In Fig. 4 the same information is presented in a rectangular plot.

B. Comments on the Model

There is no exact analytical formula that represents the atmospheric density in terms of R, although this dependence has sometimes been approximated locally by an exponential function that decreases with R. No closed-form solution of the orbit equation for quadratic drag and exponential variation of the atmospheric density with R has been found, although some interesting results in terms of perturbations and orbital elements have been found.[8–11]

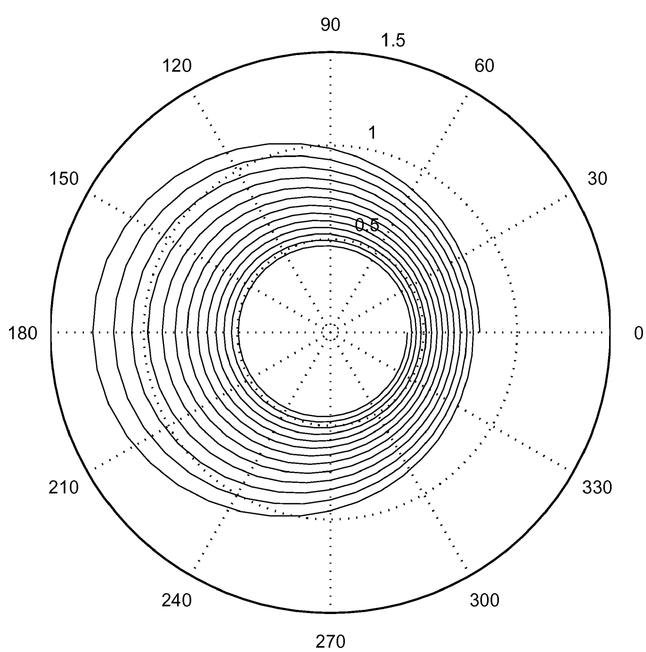

Fig. 3 Orbital decay demonstrated in a polar plot of *R*/*P* vs θ for an initial eccentricity of 0.25.

This paper presents closed-form solutions of the orbit equation for a quadratic drag model in which the atmospheric density varies inversely with the square of R for situations in which the radial speed is relatively small when compared with the tangential speed. This is only the second closed-form solution found for this class of problems. In the preceding solution the atmospheric density was assumed to vary inversely with R (Ref. 12). Although the present model is closer to an exponentially decaying density function than the preceding model, the actual improvement is miniscule.

Calculations show that when compared with an atmospheric model decreasing exponentially with altitude and a decay in altitude of 10 km, the present model represents an improvement of the order of one-tenth of 1% over the preceding model. Both models have an error of the order of 10% when compared with an exponential model. This is because of the very small relative change in R and represents one of the basic weaknesses of the models. These models come very close to quadratic drag with constant density. Interestingly enough, a constant-density quadratic drag model is not amenable to closed-form solutions, as these are. For a decay in altitude of 1 km, the error is reduced to approximately 1%. For this reason the primary usefulness of this work might be limited to arcs of orbits that are nearly circular and of high enough altitude that the decay caused by drag is not more than a few meters per revolution.

Neither model is accurate over large changes in altitude. For that matter, α cannot be regarded constant over large changes in altitude anyway. For this reason one might subdivide the altitude interval into smaller intervals and reinitialize α over each subinterval. The results of this study should therefore be viewed as local solutions. For this reason Figs. 3 and 4, which depict global solutions from Eq. (49), become increasingly inaccurate as θ increases. Of course the accuracy and usefulness of this work is much greater for near-circular orbits of very high altitude. Far more accuracy would be obtained for an initial eccentricity of 0.05, but the initial eccentricity of 0.25 used in this example demonstrates the qualitative nature of the solution. This study and the previous one[12] do show that local closed-form solutions can be found in two examples where the drag varies with the square of the magnitude of the velocity. The authors have continued the search for closed-form solutions for other quadratic drag models and expect to present a more accurate model in the future. Although this work does not consider the oblateness of the Earth or planet, it would be more accurate for near-equatorial orbits.

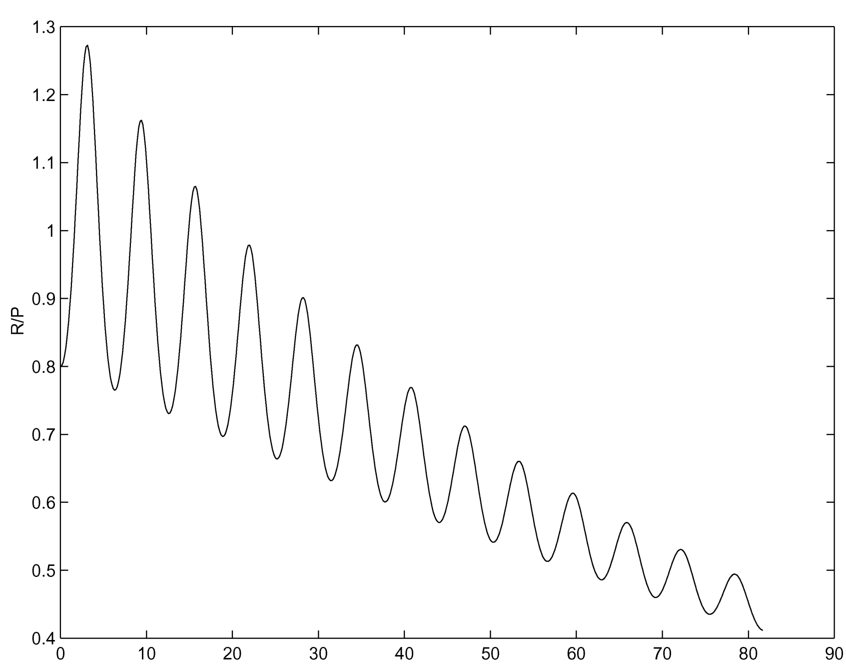

Fig. 4 Orbital decay demonstrated in a Cartesian plot of *R*/*P* vs θ.

V. Conclusions

In this paper we examined the motion of a particle in a central force field with drag that varies with a power of the magnitude of the velocity. In the case where the drag is quadratic and the radial speed is relatively small, the orbit equation can be written in a form in which the drag forces affect only one symbol in this equation. This allows it to be transformed to an ordinary differential equation.

Under these conditions, closed-form solutions for the orbit can be found for Newtonian gravitation, and the time of flight can be expressed as an integral if the atmospheric density varies inversely with the square of the radial distance from the center of attraction. The simplicity of the solutions enables one to see without difficulty the effect of drag on the orbit.

This is only the second example of closed-form solutions that have been found for this type of problem. For the earlier closed-form solution found by the authors,[12] the atmospheric density was assumed to vary inversely with the distance from the center of attraction. The improvement in accuracy of the new solutions over the previous ones are miniscule. However, the closed-form solutions of nearly circular arcs are very accurate if the drag is sufficiently small. For example, if a satellite is in a sufficiently high nearly circular orbit and loses 100 m of altitude during one revolution because of drag, the error of the atmospheric density model is of the order of one-tenth of 1%. Unfortunately, the error in the atmospheric density model increases with increasing drag and decreasing altitude. Although a more accurate representation of the atmospheric density is desirable, heretofore none has led to a closed-form solution of the orbit equation.

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